Abstract. We propose an asymptotically unbiased and consistent estimate of the bispectrum of a stationary continuous-time process $X = \{X(t)\}_{t \in \mathbb{R}}$. The estimate is constructed from observations obtained by a random sampling of the time by $\{X(\tau_k)\}_{k \in \mathbb{Z}}$, where $\{\tau_k\}_{k \in \mathbb{Z}}$ is a sequence of real random variables, generated from a Poisson counting process. Moreover, we establish the asymptotic normality of the constructed estimate.

Keywords: Periodogram, Cumulants, Quadratic-mean consistency, Bispectral density, Point process.

1 Introduction

The idea of constructing the Fourier transforms of high order cumulants was suggested by Kolmogorov, and polyspectra were introduced in [Shiryaev, 1962]. In [Brillinger, 1965] and [Brillinger and Rosenblatt, 1967] authors gave a comprehensive treatment of the theoretical properties of polyspectra, and have discussed also the estimation of polyspectra from sample records (these estimation procedures are based on a generalization of the window technique applied to products of the finite Fourier transform of the data).

Bispectra was discussed in [Tukey, 1959] and [Akaike, 1966], and an application of the bispectral analysis to the study of ocean waves is given in [Hasselmann et al., 1963], to tides in [Cartwright, 1968], and to turbulence in [Lii et al., 1976] and in [Helland et al., 1979].

These Fourier transforms -or rather, the Fourier transforms of the corresponding high order cumulants- are called polyspectra, and are defined formally as follows.

Let $\{X(t)\}_{t \in \mathbb{R}}$ be a weakly stationary process up to order $k$, and let the real number $C^{(k)}_X(s_1, s_2, \ldots, s_{k-1})$ denotes the joint cumulant of order $k$ of the set of random variables $\{X(t), X(t+s_1), \ldots, X(t+s_{k-1})\}$, i.e. $C^{(k)}_X(s_1, s_2, \ldots, s_{k-1})$ is the coefficient of $(z_1, \ldots, z_k)$ in the expansion of the cumulant generating function

$$\kappa(z_1, \ldots, z_k) = \ln \mathbb{E}\{\exp(z_1X(t) + z_2X(t+s_1) + \cdots + z_kX(t+s_{k-1}))\}.$$  

(note that, by the stationarity condition, $C^{(k)}_X(s_1, \ldots, s_{k-1})$ does not depend on $t$).
The outstanding property of polyspectra is that all polyspectra of order higher than two vanish when \( \{X(t)\}_{t \in \mathbb{R}} \) is a Gaussian process. This follows immediately from the well-known property that all joint cumulants of order higher than two vanish for multivariate Gaussian distributions. Hence, the bispectrum, trispectrum, and all higher order polyspectra are identically zero if \( \{X(t)\}_{t \in \mathbb{R}} \) is Gaussian, and these higher order spectra may thus be regarded as measures of the departure of the process from Gaussianity. The main aim of this paper is then to construct an estimate of the bispectrum of continuous-time stochastic process from a random sampling and to study its asymptotic properties, namely its mean-square convergence and its asymptotic normality.

In Section 2, we give some preliminaries on the time-sampling technique adopted here. Section 3 is concerned with the construction of the bispectrum estimate and the main results of its asymptotic properties. The last section is devoted to the proofs.

2 Preliminaries

Let \( X = \{X(t)\}_{t \in \mathbb{R}} \) be a 4th order stationary process, with mean zero and continuous integrable covariance function. From [Karr, 1991], there exists a counting process \( \mathcal{N} \), independent of \( X \), which is associated to a sequence \( \{\tau_k\}_{k \in \mathbb{Z}} \) of random variables taking their values in \( \mathbb{R} \). The process \( \mathcal{N} \) is defined by:

\[
\mathcal{N}(A, \omega) = \sum_{k \in \mathbb{Z}} 1 \text{ for } (A, \omega) \in B_\mathbb{R} \times \Omega,
\]

where \( B_\mathbb{R} \) is the Borel \( \sigma \)-algebra and \( \mathcal{N}(A, \omega) \) is the number of \( \tau_k \)'s belonging to \( A \).

We assume that, for every set \( A \) in \( B_\mathbb{R} \), the random variable \( \mathcal{N}(A) \) has a Poisson distribution \( P(\Lambda(A)) \), where \( \Lambda(A) = \beta \mu(A) \) and \( \mu \) is the Lebesgue measure on \( \mathbb{R} \) and \( \beta \) denotes the mean intensity which is assumed to be known.

We consider the sample process \( Z = \{Z(t)\}_{t \in \mathbb{R}} \) constructed from the sequence \( \{X(\tau_n)\}_{n \in \mathbb{N}} \) and the counting process \( \mathcal{N}(t) \) as follows.

**Definition 1** The sample process \( Z \) is defined by:

\[
Z(A) = \int_A X(t) \mathcal{N}(dt) = \sum_{k \in \mathbb{Z}} X(\tau_k) \mathbb{1}_A(\tau_k) = \sum_{\tau_k \in A} X(\tau_k), \quad \forall A \in B_\mathbb{R}.
\]

The process \( Z \) is also called increment-process and can be written as:

\[
Z(t) = \int_0^t X(s) \, d\mathcal{N}(s)
\]

or in the differential representation: \( dZ(t) = X(t) \, d\mathcal{N}(t) \), which proves that \( Z \) is a stationary process and that its covariance function \( R_Z \) is such that:

\[
R_Z(du) = R_X(u) \left( \beta \delta(u) + \beta^2 \right) \, du.
\]

Denote, respectively, by \( \phi^{(2)}_X \) and \( \phi^{(2)}_Z \) the spectral densities of the process \( X \) of the process \( Z \) and the increment process \( Z \).

If \( R_X \) and its Fourier transform \( \mathcal{F}R_X \) are absolutely integrable then \( \phi^{(2)}_Z \) exists, is bounded, uniformly continuous and is given by

\[
\phi^{(2)}_Z(\lambda) = \beta^2 \phi^{(2)}_X(\lambda) + \frac{\beta}{2\pi} R_X(0), \quad \forall \lambda \in \mathbb{R}
\]
Thus, the estimate of $\phi_X^{(2)}$ can be deduced from the estimate of $\phi_Z^{(2)}$ and $R_X(0)$ (Cf [Lii and Helland, 1982], [Lii and Masry, 1994], [Monsan and Rachdi, 1999] and [Gallego and Ruiz, 2000]).

3 Bispectrum estimation and its asymptotic properties

Let us denote by $\phi_X^{(3)}$ and $\phi_Z^{(3)}$ the respective bispectrum of $X$ and $Z$. Set

$$C_X^{(3)}(u_1, u_2) = \text{cum} \{X(u_1 + t), X(u_2 + t), X(t)\} = \mathbb{E} \{X(u_1 + t) X(u_2 + t) X(t)\},$$

and

$$dC_Z^{(3)}(u_1, u_2) = \text{cum} \{dZ(u_1 + t), dZ(u_2 + t), dZ(t)\}.$$

then $\phi_X^{(3)}$ and $\phi_Z^{(3)}$ are defined by

$$\phi_X^{(3)}(\lambda_1, \lambda_2) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} C_X^{(3)}(u_1, u_2) \exp(-i(\lambda_1 u_1 + \lambda_2 u_2)) \, du_1 \, du_2,$$

$$\phi_Z^{(3)}(\lambda_1, \lambda_2) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \exp(-i(\lambda_1 u_1 + \lambda_2 u_2)) \, dC_Z^{(3)}(u_1, u_2).$$

As in the previous section, using the independence between $X$ and $Z$, we establish a relationship between $dC_Z^{(3)}$ and $C_X^{(3)}$. First, we have

$$dC_Z^{(3)}(u_1, u_2) = C_X^{(3)}(u_1, u_2) [\beta \delta(u_1) \delta(u_2) + \beta^2 \delta(u_1) + \beta^2 \delta(u_2) + \beta^2 \delta(u_1 - u_2) + \beta^3] \, du_1 \, du_2,$$

thus, if we define by:

$$\psi(\lambda) = \left(\int_{-\infty}^{\infty} \exp(i \lambda u) C_X^{(3)}(u_1, u_1) \, du_1 + C_X^{(3)}(0, 0)\right) \beta^2 / (2\pi)^2$$

we can write

$$\phi_Z^{(3)}(\lambda_1, \lambda_2) = \beta^2 \phi_X^{(3)}(\lambda_1, \lambda_2) - \frac{2\beta}{(2\pi)^2} C_X^{(3)}(0, 0) + \psi(\lambda_1) + \psi(\lambda_2) + \psi(-\lambda_1 - \lambda_2)$$

and

$$\phi_X^{(3)}(\lambda_1, \lambda_2) = \frac{1}{\beta^3} \left[\phi_Z^{(3)}(\lambda_1, \lambda_2) + \frac{2\beta}{(2\pi)^2} C_X^{(3)}(0, 0) - \psi(\lambda_1) - \psi(\lambda_2) - \psi(-\lambda_1 - \lambda_2)\right].$$

Equation (1) is the relationship which allows to estimate $\phi_X^{(3)}$ from discrete data as follows. Given the observations $\{X(\tau_k)\}_{k=1}^{N(T)}$, $T > 0$, where $N(T)$ is the number of points $\{\tau_k\}$ falling in $[0, T]$, we estimate the bispectrum $\phi_X^{(3)}$ by estimating the bispectrum of $Z$, the function $\psi$ and the constant $C_X^{(3)}(0, 0)$. For this, let us denote by $\hat{\phi}_X^{(3)}$, $\hat{\phi}_Z^{(3)}$, $\hat{\psi}$ and $\hat{\psi}_T$ the respective estimates of $\phi_X^{(3)}$, $\phi_Z^{(3)}$ and $\psi$. Consider the three dimensional spectral window $W$ and the bandwidth $b_T$, which verify the following assumptions.

Assumptions 1
(i) \( W \in L^1 \cap L^{\infty}(\mathbb{R}^3) \) is a positive function such that:
\[
\int_{\mathbb{R}^2} |W(u_1, u_2, -u_1 - u_2)| \, du_1 \, du_2 = 1
\]

(ii) \( |W(u_1, u_2, -u_1 - u_2)| \) and \( |\frac{\partial W(u_1, u_2, -u_1 - u_2)}{\partial u_j}| \leq C (1 + ||(u_1, u_2)||_2)^{-2-\varepsilon} \),
\( j = 1, 2 \) where \( C \) and \( \varepsilon \) are two positive real numbers, and \( ||(u_1, u_2)||_2 \) denotes the euclidean norm of \((u_1, u_2)\).

(iii) \((b_T)_{T \in \mathbb{R}}\) is a sequence of positive real numbers such that:
\[
b_T \to +\infty \text{ and } \frac{b_T^2}{T} \to 0 \text{ as } T \to +\infty
\]

In order to construct \( \hat{\phi}_{z,T}^{(3)} \), we set: \( W_T(u_1, u_2, u_3) = b_T^2 W(b_T u_1, b_T u_2, b_T u_3) \).

The periodogram (empirical estimate of \( \hat{\phi}_{z,T}^{(3)} \)) is defined by
\[
\hat{I}_T(\lambda_1, \lambda_2) = \frac{1}{(2\pi)^2 T} \hat{d}_{z,T}(\lambda_1) \hat{d}_{z,T}(\lambda_2) \hat{d}_{z,T}(-\lambda_1 - \lambda_2),
\]
where \( \hat{d}_{z,T}(\lambda) = \sum_{k=1}^{N(T)} X(\tau_k) \exp(-i \lambda \tau_k) \), is the finite Fourier transform of the observations \( \{X(\tau_k)\}_{k=1}^{N(T)} \).

In order to obtain a consistent estimate of \( \hat{\phi}_{z,T}^{(3)} \), we smooth \( \hat{I}_T \) by \( W_T \), which yields to:
\[
\hat{\phi}_{z,T}^{(3)}(\lambda_1, \lambda_2) = \left(\frac{1}{2\pi T}\right)^2 \sum_{i,j=-\infty}^{+\infty} W_T(\lambda_1 - \omega_i, \lambda_2 - \omega_j, -\lambda_1 - \lambda_2 - \omega_i + j) \hat{I}_T(\omega_i, \omega_j)
\]

where \( \omega_j = 2\pi j / T \) for \( j \in \mathbb{Z} \) denotes the Fourier frequencies.

**Assumptions 2**

\( R_X \) and \( C_X^{(3)} \) are absolutely integrable and for all \( k = 1, \ldots, 5 \)
\[
\int_{\mathbb{R}^k} ||u||_1 |\phi_X^{(k+1)}(u)| \, du < +\infty,
\]
where \( u = (u_1, \ldots, u_k) \) and \( ||u||_1 = \sum_{j=1}^{k} |u_j| \), where \( \phi_X^{(k)}(u) \) denotes the \( k \)-th derivative of \( \phi_X(u) \).

We have that under Assumptions 2, the measure \( dC_Z^{(3)} \) is integrable and
\[
\int_{\mathbb{R}^k} ||u||_1 |dC_Z^{(k+1)}(u)| < +\infty, \text{ for } k = 1, \ldots, 5
\]
from which we infer the following properties that are useful to establish the asymptotic behavior of \( \hat{\phi}_{z,T}^{(3)}(\lambda_1, \lambda_2) \) (Cf Theorem 1). For this aim, the following proposition gives the asymptotic behavior of the bias and the covariance of the periodogram \( \hat{I}_T \).
Proposition 1 Let \( \lambda_1, \lambda_2, \mu_1 \) and \( \mu_2 \) be any real numbers, then
\[
\mathbb{E}\{\hat{I}_\tau(\lambda_1, \lambda_2)\} = \phi_2^{(3)}(\lambda_1, \lambda_2) + O(1/T)
\]
where \( O(1/T) \) is uniform in \( \lambda_1 \) and \( \lambda_2 \). Moreover, the covariance is
\[
\frac{1}{T} \text{cov}\{\hat{I}_\tau(\lambda_1, \lambda_2), \hat{I}_\tau(\mu_1, \mu_2)\} = \frac{1}{2\pi} \sum_{\rho \in S_3} \prod_{i=1}^{3} \Delta_T(\lambda_i + \mu_{p(i)}) \phi_2^{(3)}(\lambda_i) + O(1/T)
\]
where \( \lambda_3 = -(\lambda_1 + \lambda_2), \mu_3 = -(\mu_1 + \mu_2), S_3 \) is the set of all permutations of \( \{1, 2, 3\} \) and \( \Delta_T(\lambda) = \sin(T\lambda/2)/(T\lambda/2) \).

Theorem 1 If both assumptions 1 and 2 are satisfied, then the bias and the covariance of \( \hat{\phi}_2^{(3)}(\cdot) \) are given by
\[
\mathbb{E}\{\hat{\phi}_2^{(3)}(\lambda_1, \lambda_2)\} = \phi_2^{(3)}(\lambda_1, \lambda_2) + O(1/T)
\]
and
\[
\lim_{T \to +\infty} \frac{T}{\hat{b}_T} \text{cov}\{\hat{\phi}_2^{(3)}(\lambda_1, \lambda_2), \hat{\phi}_2^{(3)}(\mu_1, \mu_2)\}
\]
\[
= 2\pi \prod_{i=1}^{3} \phi_2^{(3)}(\lambda_i) \left( \sum_{\rho \in S_3} \prod_{i=1}^{3} \delta(\lambda_i - \mu_{p(i)}) \right) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W^2(u_1, u_2, -u_1 - u_2) \, du_1 \, du_2.
\]

In order to estimate the function \( \psi \), we propose the following estimate:
\[
\hat{\psi}_T(\lambda) = \frac{1}{(2\pi)^2 T} \int_{-\infty}^{+\infty} W_T(\lambda - u) \hat{D}_{X,T}(u) \hat{D}_{X,T}(-u) \, du,
\]
where \( \hat{D}_{X,T}(\lambda) = \sum_{j=1}^{N(T)} \exp(-i\lambda \tau_j) X^2(\tau_j) \). Then, the asymptotic behavior of this estimate is studied in the following proposition.

Proposition 2 Under Assumptions 1 and 2, we have
\[
\mathbb{E}\{\hat{\psi}_T(\lambda)\} = \psi(\lambda) + O\left(\frac{1}{T}\right),
\]
and
\[
\text{cov}\{\hat{\psi}_T(\lambda), \hat{\psi}_T(\mu)\} = C_3 \delta(\lambda)\delta(\mu) D_T^2(0) \int \mathbb{R}^2 D_1(x) D_1(y) D_1(-x - y) \, dx \, dy + \frac{\beta^2}{2\pi} (\delta(\lambda) \delta(\mu)) W(0) + O\left(\frac{1}{b_T}\right)
\]
For the term \( C_3^{(3)}(0,0)\beta/(2\pi)^2 \), we propose the following estimate:
\[
\hat{C}_T = \frac{1}{(2\pi)^2 T} \sum_{j=1}^{N(T)} X^3(\tau_j) = \frac{1}{(2\pi)^2 T} \int_0^T X^3(t) \, dN(t)
\]
for which the asymptotic properties are given in the following proposition.
Proposition 3 Under Assumptions 1 and 2, we have
\[ \mathbb{E}\{\hat{C}_T\} = \frac{\beta}{(2\pi)^2} c_X^{(3)}(0,0) \] and \[ \text{var}\{\hat{C}_T\} = O\left(\frac{1}{T}\right) \]

Finally, in order to estimate the bispectrum of \( X \), we propose the following estimate,

\[ \hat{\phi}^{(3)}(X,T)(\lambda_1,\lambda_2) = \frac{1}{\beta^3} \left[ \hat{\phi}^{(3)}(Z,T)(\lambda_1,\lambda_2) \right. - \hat{\psi}_T(\lambda_1 + \lambda_2) - \hat{\psi}_T(\lambda_1) - \hat{\psi}_T(\lambda_2) + 2 \hat{C}_T \] (4)

and its asymptotic properties are given in the following theorems.

Theorem 2 Under Assumptions 1 and 2, the bispectrum estimate defined by equation (4) is asymptotically mean square convergent. Moreover, its asymptotic integrated mean squared error is given by:

\[ \int \left( \hat{\phi}^{(3)}(X,T)(u) - \phi^{(3)}(X)(u) \right)^2 du = O\left(\frac{\beta^2}{T}\right) \]

In the following theorem, we establish the asymptotic distribution of the bispectrum estimate.

Theorem 3 For an integer \( k \geq 3 \), suppose that \( X \) is stationary up to order \( k \) and the \( k \)th order cumulant of \( X \) is absolutely integrable. Let \( (\lambda_{1,1}, \lambda_{2,1}, \ldots, \lambda_{1,r}, \lambda_{2,r}) \) be \( r \) couples of real numbers. Then under assumptions of Theorem 2, the standardized variables

\[ \left\{ \sqrt{T/\beta^2} \left[ \hat{\phi}^{(3)}(X,T)(\lambda_{1,i},\lambda_{2,i}) - \mathbb{E}\{\hat{\phi}^{(3)}(X,T)(\lambda_{1,i},\lambda_{2,i})\} \right] \right\}_{i=1}^r \]

are jointly asymptotically normally distributed with mean 0 and covariances may be obtained easily from Theorem 1 and Propositions 2 and 3.

4 Proofs

In order to save space in this paper, the proofs of the previous results were summarized. For more details, the readers are advised to contact the authors.

Proof of Proposition 1. Let us begin by computing the bias of \( \hat{I}_r \). for this aim, by (2), we have that

\[ \mathbb{E}\{\hat{I}_r(\lambda_1,\lambda_2)\} = \frac{1}{(2\pi)^2T} \left( TD_r(0) (2\pi)^2 \phi^{(3)}_X(\lambda_1,\lambda_2) + O(1) \right) = \phi^{(3)}_X(\lambda_1,\lambda_2) + O(1/T), \]

where \( O(1/T) \) is bounded uniformly in \( \lambda_1 \) and \( \lambda_2 \), and where

\[ D_j(\lambda) = \int_0^\lambda \exp(-i \lambda t) dt \text{ and } D_T(\lambda) = \exp\left(-i \lambda T/2\right) \sin(T \lambda/2)/(\lambda/2) = TD_1(T \lambda) \]

To compute \( \text{cov}\{\hat{I}_r(\lambda_1,\lambda_2),\hat{I}_r(\mu_1,\mu_2)\} \), we use the following property :

\[ \mathbb{E}\{Y_1 \ldots Y_k Y_{k+1} \ldots Y_{2k}\} - \mathbb{E}\{Y_1 \ldots Y_k\} \mathbb{E}\{Y_{k+1} \ldots Y_{2k}\} = \sum_{\nu} \prod_{\nu_i \in \nu} \text{cum}\{Y_{j}, j \in \nu_i\} \] (5)
where the sum $\sum_{\nu}$ is extended over all the indecomposable partitions of \{1, \ldots, 2k\}. Let $\alpha_i = \lambda_i$ for $i = 1, 2, 3$ and $\alpha_i = \mu_i$ for $i = 3, 4, 5$. Then

$$T^{-1} \text{cov}\{\hat{I}_T(\lambda_1, \lambda_2), \hat{I}_T(\mu_1, \mu_2)\} = \frac{1}{(2\pi)^4 T^3} \left( \mathbb{E} \prod_{i=1}^{6} \hat{d}_{Z,T}(\alpha_i) - \mathbb{E} \prod_{i=1}^{3} \hat{d}_{Z,T}(\alpha_i) \mathbb{E} \prod_{i=4}^{6} \hat{d}_{Z,T}(\alpha_i) \right)$$

From [Brillinger, 1969] and (5), we get

$$T^{-1} \text{cov}\{\hat{I}_T(\lambda_1, \lambda_2), \hat{I}_T(\mu_1, \mu_2)\} = \frac{1}{(2\pi)^4 T^3} \sum_{\nu} \prod_{i=1}^{l} ((2\pi)^{\nu_i-1} D_{\nu} \left( \sum_{j \in \nu_i} \phi_{z}^{(\nu_i)}(\alpha_j, j \in \nu'_i) \right) + O(1)), \quad (6)$$

where $\nu$ is an indecomposable partition, $\nu'_i$ is a subpartition such that $|\nu'_i| = |\nu_i| - 1$ where $|\nu_i|$ denotes the number of elements of $\nu_i$, and $\phi_{z}^{(\nu_i)}$ is the spectral density of order $|\nu_i|$ of $Z$.

We have that $D_{\nu}$ and $\phi_{z}^{(\nu_i)}$ for $|\nu_i| \leq 6$ are bounded, then the maximum order of magnitude of the quantity (6) is $O(T^{-4})$. That is, for $l = 1$ the order of magnitude is $O(T^{-4})$, for $l = 2$ the order of magnitude is $O(T^{-1})$ and for $l = 3$ the partitions are composed of the pairs $\{\lambda_1, \mu_{p(1)}\}, \{\lambda_1, \mu_{p(2)}\}, \{\lambda_1, \mu_{p(3)}\}$ where $p \in S_3$, and if $l > 3$ then there exists a unique set in the partition and thus the cumulant vanish. Thus

$$T^{-1} \text{cov}\left\{\hat{I}_T(\lambda_1, \lambda_2), \hat{I}_T(\mu_1, \mu_2)\right\} = \frac{1}{2\pi} \sum_{p \in S_3} \prod_{i=1}^{3} D_{1} \left( T(\lambda_i + \mu_{p(i)}) \right) \phi_{z}^{(2)}(\lambda_i) + O \left( \frac{1}{T} \right)$$

$$= \frac{1}{2\pi} \sum_{p \in S_3} \prod_{i=1}^{3} \hat{d}_{T} \left( \lambda_i + \mu_{p(i)} \right) \phi_{z}^{(2)}(\lambda_i) + O \left( \frac{1}{T} \right).$$

**Proof of Theorem 1.** By some simple computations, from Proposition 1, we obtain

$$\mathbb{E}\{\phi_{z,T}^{(3)}(\lambda_1, \lambda_2)\} = \frac{(2\pi)^2}{T^3} \sum_{i,j=-\infty}^{k=\infty} W_T \left( \lambda_1 - \omega_i, \lambda_2 - \omega_j, -\lambda_1 - \lambda_2 - \omega_{(i+j)} \right) \phi_{z}^{(3)}(\omega_i, \omega_j) \quad (7)$$

$$+ O \left( \frac{(2\pi)^2}{T^3} \right) \sum_{i,j=-\infty}^{k=\infty} W_T \left( \lambda_1 - \omega_i, \lambda_2 - \omega_j, -\lambda_1 - \lambda_2 - \omega_{(i+j)} \right). \quad (8)$$

We have to study both series (7) and (8). In order to establish the asymptotic behavior of (7), we consider the finite sum (for $M \in \mathbb{N}$)

$$\sum_{i,j=-M}^{M} W_T \left( \lambda_1 - \omega_i, \lambda_2 - \omega_j, -\lambda_1 - \lambda_2 - \omega_{(i+j)} \right) \phi_{z}^{(3)}(\omega_i, \omega_j)$$
\[
\left( \frac{2\pi b_T}{T} \right)^2 \sum_{i,j=0}^{2M} W\left( -b_T \lambda_1 - b_T \omega_{M-i}, -b_T \lambda_2 - b_T \omega_{M-j}, b_T (\lambda_1 + \lambda_2) - b_T \omega_{-2M+i+j} \right) \\
\times g\left( -b_T \lambda_1 - b_T \omega_{M-i}, -b_T \lambda_2 - b_T \omega_{M-j} \right)
\]

(9)

where \( g \) denotes the function \( g(x, y) = \phi_x^{(3)}(x/b_T + \lambda_1, y/b_T + \lambda_2) \).

On the other hand, we have that: when \( A_j, B_j, h_j \) and \( N_j \) be real numbers such that \( A_j \leq B_j, h_j > 0 \) and \( N_j = B_j - A_j/h_j \) for \( j = 1, \ldots, k \), then

\[
\prod_{i=1}^{k} \sum_{i_1=0}^{N_j-1} \cdots \sum_{i_{k-1}=0}^{N_j-1} g(A_1 + n_1h_1, \ldots, A_k + n_kh_k) = \int \prod_{i=1}^{k} g(v) \, dv + O \left( \frac{b_T}{T} \right)
\]

where \( |R| \leq C \sum_{j=1}^{k} h_j \) with \( C \) is a positive constant uniform in \( k \). Thus, by applying (10), the term (9) becomes

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W(u_1, u_2, -u_1 - u_2) \phi_x^{(3)}(\frac{u_1}{b_T} + \lambda_1, \frac{u_2}{b_T} + \lambda_2) \, du_1 \, du_2 + O \left( \frac{b_T}{T} \right)
\]

since \( R \) verifies \( |R| \leq 4\pi C b_T/T \). Thus, when \( M \to +\infty \), (9) becomes

\[
O\left( \frac{1}{T} \right) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left| W_T\left( \lambda_1 - x, \lambda_2 - y, -\lambda_1 - \lambda_2 + x + y \right) \right| \, dx \, dy = O\left( \frac{1}{T} \right),
\]

thus

\[
\mathbb{E}\{ \hat{\phi}_x^{(3)}(\lambda_1, \lambda_2) \} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W\left( u_1, u_2, -u_1 - u_2 \right) \phi_x^{(3)}\left( \frac{u_1}{b_T} + \lambda_1, \frac{u_2}{b_T} + \lambda_2 \right) \, du_1 \, du_2 + \left( \frac{b_T}{T} \right)
\]

Since \( \phi_x^{(3)} \) is bounded, \( W \) is absolutely integrable and \( b_T \to +\infty \), the result is obtained by a simple application of the dominated convergence theorem.

Now, we are concerned with the consistency of \( \hat{\phi}_x^{(3)} \). For this, from Proposition 1, the general term of the covariance \( \text{cov} \left\{ \hat{\phi}_x^{(3)}(\lambda_1, \lambda_2), \hat{\phi}_x^{(3)}(\mu_1, \mu_2) \right\} \) is:

\[
O\left( \frac{(2\pi b_T)^{6.1}}{T^3} \right) \sum_{s_1+s_2+s_3=-\infty}^{+\infty} \sum_{r_1+r_2+r_3=-\infty}^{+\infty} W_T(\lambda_1 - \omega_{s_1}, \lambda_2 - \omega_{s_2}, \lambda_3 - \omega_{s_3})
\]

\[
W_T(\mu_1 - \omega_{r_1}, \mu_2 - \omega_{r_2}, \mu_3 - \omega_{r_3}) \delta(\sum_{i=1}^{3} \omega_{s_i}) \delta(\sum_{i=1}^{3} \omega_{r_i}) \prod_{i=1}^{l} T \delta(\omega_{S_{i\nu_i}}) \phi_x^{(3)}(\omega_{s_i}, \alpha \in \nu_i)
\]

which may be written as the sum of two terms and we have to study the asymptotic behavior of each one.
The first term of (11) is:

$$\frac{(2\pi)^{6-p}}{T^{6-1}} \sum_{s_1,s_2,s_3} \sum_{r_1,r_2,r_3} W_T(\lambda_1 - \omega_{s_1}, \lambda_2 - \omega_{s_2}, \lambda_3 - \omega_{s_3}) W_T(\mu_1 - \omega_{r_1}, \mu_2 - \omega_{r_2}, \mu_3 - \omega_{r_3})$$

$$\times \delta(\sum_{i=1}^{3} \omega_{s_i}) \delta(\sum_{i=1}^{3} \omega_{r_i}) \prod_{i=1}^{l} T(\omega_{S_{s_i}}) \phi^{(\nu)}_2(\omega_{\alpha}, \alpha \in \nu_i).$$

(12)

Notice that this sum contains no more than \((5 - l)\) variables, but that we consider these \((5 - l)\) variables since at the end of the calculations we extend the sum to all the terms. Thus (12) may be written as

$$\left[\frac{(2\pi)^{1+1-p} b_T^{l-1}}{T}\right] \left(\frac{2\pi b_T}{T}\right)^{5-l} \sum_{s_1,s_2,s_3} \sum_{r_1,r_2,r_3} W(b_T (\lambda_1 - \omega_{s_1}), b_T (\lambda_2 - \omega_{s_2}), b_T (\lambda_3 - \omega_{s_3}))$$

$$\times W(b_T (\mu_1 - \omega_{r_1}), b_T (\mu_2 - \omega_{r_2}), b_T (\mu_3 - \omega_{r_3}) \delta(\sum_{i=1}^{3} \omega_{s_i}) \delta(\sum_{i=1}^{3} \omega_{r_i}) \prod_{i=1}^{l} T(\omega_{S_{s_i}}) \phi^{(\nu)}_2(\omega_{\alpha}, \alpha \in \nu_i).$$

(13)

Moreover, by applying (10) to the general term of the sum (13), we get

$$\int_{-\infty}^{+\infty} W(u_1, u_2, u_3) W(v_1, v_2, v_3) \delta(u_1 + u_2 + u_3) \delta(v_1 + v_2 + v_3)$$

$$\times \prod_{i=1}^{l} \delta(S_{\gamma_i} + S_{\eta_i}) \phi^{(\nu)}_2 \left(\alpha_j + \frac{\beta_j}{b_T}, \alpha_j \in \gamma_i', \beta_j \in \eta_i' \right) du_1 dv_1,$$

where \(\gamma_i \subset \{\lambda_1, \lambda_2, \lambda_3, -\mu_1, -\mu_2, -\mu_3\}\) and \(\eta_i \subset \{u_1, u_2, u_3, -v_1, -v_2, -v_3\}\) analogous to \(\nu_i\) element of the partition \(\nu, S_{\gamma_i} = \sum_{x \in \gamma_i} x, S_{\eta_i} = \sum_{x \in \eta_i} x,\)

and \(\gamma_i' \subset \gamma_i\) such that \(|\gamma_i'| = |\gamma_i| - 1\), and \(\eta_i'\) is defined in a similar way.

Remark that, if \(l > 3\), the studied term vanishes since there exists an element of the partition which is a single set. But if \(l \leq 3\), the main terms are given for \(l = 3\). We conclude that the first member of the general term is:

$$\frac{2\pi b_T^2}{T} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \prod_{i=1}^{l} \delta(S_{\gamma_i} + S_{\eta_i}) \phi^{(\nu)}_2 \left(\alpha_i + \frac{\beta_i}{b_T}, \alpha_i \in \gamma_i', \beta_i \in \eta_i' \right) du_1 dv_1 dv_1 du_2 dv_2 + O\left(\frac{b_T^3}{T^2}\right).$$

The second term of (11) may be bounded by:

$$O\left(\frac{(2\pi b_T)^4}{T}\right) \sum_{s_1,s_2,s_3} \sum_{r_1,r_2,r_3} |W(b_T (\lambda_1 - \omega_{s_1}), b_T (\lambda_2 - \omega_{s_2}), b_T (\lambda_3 - \omega_{s_3}))|$$

$$\times |W(b_T (\mu_1 - \omega_{r_1}), b_T (\mu_2 - \omega_{r_2}), b_T (\mu_3 - \omega_{r_3})) \delta(\sum_{i=1}^{3} \omega_{s_i}) \delta(\sum_{i=1}^{3} \omega_{r_i})|$$
Moreover, on the other hand, from (10), we have
\[ \int \ldots \int_{-\infty}^{+\infty} |W(u_1, u_2, -u_1 - u_2)W(v_1, v_2, -v_1 - v_2)| \, du_1 \, du_2 \, dv_1 \, dv_2 + O \left( \frac{b_T}{T} \right). \]
Thus the second term behaves as \( o \left( \frac{b_T^2}{T} \right) \). Since \( W \) is symmetric and absolutely integrable, the result is then obtained by a simple application of the dominated convergence theorem.

**Proof of Proposition 2.** From (10), we have by a classical calculation that
\[ E\{\hat{\psi}_T(\lambda)\} = \int_{-\infty}^{+\infty} W(u)\psi \left( \lambda - \frac{u}{b_T} \right) \, du + O \left( \frac{1}{T} \right). \]
As \( W \) is absolutely integrable and \( \psi \) is bounded, the result (3) is obtained by a simple application of the dominated convergence theorem.

On the other hand, from (10), we have
\[ \text{cov} \left\{ \hat{\psi}_T(\lambda), \hat{\psi}_T(\mu) \right\} \]
\[ = \frac{\beta^2}{2\pi T} \left( W(cT(\lambda) + C_1W_T(\mu)) + O \left( \frac{b_T}{T} \right) + O \left( \frac{1}{T} \right) \right) \]
\[ + \frac{\beta^2}{2\pi T} \int_{-\infty}^{+\infty} (C_2W_T(\mu + u) + C_4W_T(\lambda - u)) \, du \]
\[ + C_5 T \int_{-\infty}^{+\infty} W_T(\lambda - u) W_T(\mu - u) D_1(T(u)) D_1(T(v)) D_1(-T(u + v)) \, du \, dv \]

Then likewise by the dominated convergence theorem, we obtain:
\[ \int_{-\infty}^{+\infty} W(x)W(b_T (\mu - \lambda) + x) \, dx \rightarrow \delta(\lambda - \mu) \int_{-\infty}^{+\infty} D_1^2(x) \, dx \]
and
\[ \int_{-\infty}^{+\infty} W(x)W(b_T (\mu + \lambda) + x) \, dx \rightarrow \delta(\lambda + \mu) \int_{-\infty}^{+\infty} D_1^2(x) \, dx \text{ as } T \rightarrow +\infty. \]

Moreover, \( \int_{-\infty}^{+\infty} W(b_T (\lambda - \frac{x}{T}))W(b_T (\mu - \frac{y}{T})) D_1(x) D_1(y) D_1(-x - y) \, dx \, dy \)
\[ \rightarrow \delta(\lambda)\delta(\mu) D_1^2(0) \int_{-\infty}^{+\infty} D_1(x) D_1(y) D_1(-x - y) \, dx \, dy \text{ as } T \rightarrow +\infty, \]
which leads to
\[ \text{cov} \left\{ \hat{\psi}_T(\lambda), \hat{\psi}_T(\mu) \right\} \]
\[ = \frac{\beta^2}{(2\pi)^2 T} \left[ 2\pi C_1 W_T(\lambda) + 2\pi C_2 W_T(\mu) + 2\pi b_T C_3 \int_{-\infty}^{+\infty} W(x)W(b_T (\mu - x) + x) \, dx \right] + \frac{\beta^2}{(2\pi)^2 T} \left[ 2\pi b_T C_3 \int_{-\infty}^{+\infty} W(x)W(b_T (\lambda - x) + x) \, dx + O \left( \frac{b_T}{T} \right) \right] + O \left( \frac{1}{T} \right) \]
\[ + C_5 \frac{b_T^2}{T} \int_{-\infty}^{+\infty} W(b_T (\lambda - \frac{x}{T}))W(b_T (\mu + \frac{y}{T})) D_1(T(x)) D_1(T(y)) D_1(-T(x + y)) \, dx \, dy \]
which completes the proof.

**Proof of Proposition 3.** The average of $\hat{C}_T$ is such that

$$E\{\hat{C}_T\} = \frac{1}{(2\pi)^2} \int_0^T E\{X^3(0)\} E\{dN(t)\} = \frac{\beta}{(2\pi)^2} C_X^{(3)}(0,0)$$

Hence, $\hat{C}_T$ is an unbiased estimate of $C_X^{(3)}(0,0)\beta/(2\pi)^2$.

For the variance, we have that

$$E\{\hat{C}_T^2\} = \frac{\beta^2}{(2\pi)^4 T^2} \int_0^T \int_0^{T-t} E\{X^3(u)X^3(0)\} du \, dt + \frac{\beta}{(2\pi)^4 T} E\{X^6(0)\}$$

which completes the proof.

**Proof of Theorem 2.** The asymptotic behavior of the bias of the estimate is directly obtained by writing

$$E\{\hat{\phi}_x^{(3)}(X,T)\} = \frac{1}{\beta^3} \left( E\{\hat{\phi}_x^{(3)}(X,T)\} - \psi(\lambda_1) - \psi(\lambda_2) - \psi(\lambda_1 + \lambda_2) + \frac{2\beta^2}{(2\pi)^2} C_X^{(3)}(0,0) \right) + O\left(\frac{b_T}{T}\right)$$

$$= \phi_x^{(3)}(\lambda_1, \lambda_2) + O\left(\frac{b_T}{T}\right).$$

To compute the cross-covariance of the estimate $\hat{\phi}_x^{(3)}(X,T)$, we use the following properties:

$$|\text{cov}\{\hat{\psi}_T(\lambda), \hat{\phi}_x^{(3)}(X,T)\}|^2 = O\left(\frac{b_T^2}{T}\right) O\left(\frac{b_T^2}{T}\right) = O\left(\frac{b_T^4}{T^2}\right)$$

$$|\text{cov}\{\hat{\psi}_T(\lambda), \hat{C}_T\}|^2 = O\left(\frac{b_T^2}{T}\right) O\left(\frac{1}{T}\right) = O\left(\frac{b_T^2}{T^2}\right)$$

$$|\text{cov}\{\hat{\phi}_x^{(3)}(X,T), \hat{C}_T\}|^2 = O\left(\frac{b_T^2}{T}\right) O\left(\frac{1}{T}\right) = O\left(\frac{b_T^2}{T^2}\right).$$

Thus, we deduce that

$$\text{cov}\{\hat{\phi}_x^{(3)}(X,T), \hat{\phi}_x^{(3)}(X,T)\} = O\left(\frac{b_T^2}{T}\right)$$

which completes the proof of the theorem.

**Proof of Theorem 3.** Notice that, for space reasons we omitted all the technical details which could be obtained from the authors. In another hand, the reader can refer to the papers [Li and Masry, 1994] and [Monsan and Rachdi, 1999] and follow the same steps in order to prove the asymptotic normality of the estimate by using the cumulants theory.
References


